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THE ANALOGUE OF WIENER SPACE WITH VALUES IN ORLICZ SPACE

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Abstract. Let M be an N-function satisfies the $\triangle_2$-condition and let O_M be the Orlicz space associated with M. Let $C(O_M)$ be the space of all continuous functions defined on the interval $[0,T]$ with values in O_M.

In this note, we define the analogue of Wiener measure m_ϕ^M on $C(O_M)$, establish the Wiener integration formulae for the cylinder functions on $C(O_M)$ and give some examples related to our formulae.

1. Introduction

It is the starting point of the study for Brownian motion that Robert Brown observed the motions of small particles in water through a microscope in 1827. Since then, Wiener had established a theory for the reasonable probability measure m_ω associated with Brownian motion, the one-dimensional Wiener measure, on the space $C_0[0,T]$ of all real-valued continuous functions on the closed bounded interval $[0,T]$ that vanish at 0 in 1923[12]. In 1965, Gross presented the theory for the abstract Wiener mesure ω on the infinite dimesional real seperable Banach spcae $\mathbb{B}$[2]. These are Gaussian measures on $C_0[0,T]$ and $\mathbb{B}$, respectively. In 1972, Rajput introduced the theory of Gaussian measures on L_p spaces, $1 \leq p \leq +\infty$[8], in 1977, Byczkowski studied the theory of the Gaussian measures on L_p spaces, $0 \leq p \leq +\infty$[1], and in 1981, Lawniczak researched the Gaussian measure on Orlicz space, which is a kind of generalization of L_p space[7].

In 1973, Kuelbs and LePage suggested the existence of non-zero stationary increment Gaussian measure $m_{\mathbb{B}}$ over paths in abstract Wiener

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space $C_0(\mathbb{B})$, the space of all $\mathbb{B}$-valued continuous functions on $[0,T]$ that vanish at 0[6], and in 1986, Jurlewicz presented the theory of the Gaussian measures on $C_0(O_M)$, the space of all O_M-valued continous functions on $[0,T]$ that vanish at 0[4]. In 1992, the authour showed the existence of $m_{\mathbb{B}}$, by the different way from Kuelbs and LePage's method, and found the Wiener integration formuala for it[9]. In 2002, the author and Dr.Im defined the analogue of Wiener measure on $C[0,T]$, associated with the Borel measure ϕ on $\mathbb{R}$[11] and the author proved the existence theorem of analogue of Wiener measure space over paths in abstract Wiener space $\mathbb{B}$, associated with the Borel measure on $\mathbb{B}$[10].

In this article, we introduce the analogue of Wiener measure m_ϕ^M on $C(O_M)$, assiciated with Borel mesure ϕ on O_M, establish the Wiener integration fomulae for cylinder functions on $C(O_M)$ and give some examples of it.

2. Preliminaries: definitions, notations and properties

In this section, we introduce some definitions and notations which are needed to understand this article.

In [5], we can find the fundamental properties of the Orlicz space.

(A) A real-valued continuous function $M(u)$ is called an N-function if it is even and satisties $\lim_{u\to\infty}\frac{M(u)}{u}=+\infty$ and $\lim_{u\to 0}\frac{M(u)}{u}=0$, equivalent to it admits of the representation $M(u)=\int_0^{|u|}p(t)dt$ where the function $p(t)$ is right-continuous for $t\geq 0$, positive for $t>0$ and non-decreasaing which satisfies the condition $p(0)=0$ and $\lim_{t\to+\infty}p(t)=+\infty$.

(B) For right-continuous for $t\geq 0$, positive for $t>0$ and non-decreasing function p, having the properties $p(0)=0$ and $\lim_{t\to+\infty}p(t)=+\infty$, let $q(s)=\sup_{p(t)\leq s}t$ for $s\geq 0$. Then q is right-continuous for $s\geq 0$, positive for $s>0$ and non-decreasing, $q(0)=0$ and $\lim_{s\to+\infty}q(s)=+\infty$. Let $N(v)=\int_0^{|v|}q(s)ds$. Then N is an N-function. Here, we say that M and N are mutually complimentary N-functions

(C) We say that an N-function M satisties the $\triangle_2$-condition if there are two constants u_0 and k such that for $u\geq u_0$, $M(2u)\leq kM(u)$ and we say that an N-function M satisfies the $\triangle_a$-condition if $\overline{\lim}_{u\to+\infty}\frac{M(u^2)}{M(u)}<+\infty$.

Remark 2.1.

(1) *If an N-funtion M satisties the $\triangle_2$-condition then there are two constants α and c with $\alpha > 1$ and $c > 0$ such that $M(u) \leq c|u|^{\alpha}$ for sufficiently large value of u.*
(2) *When $\alpha > 1$ and $a > 0$, $M(u) = a|u|^{\alpha}$ and $M(u) = |u|^{\alpha}(\ln|u| + 1)$ satisfy the $\triangle_2$-condition and $M(u) = e^{|u|} - |u| - 1$ doesn't satisfy the $\triangle_2$-condition.*

(D) For an N-function M and for a measurable function $u : [0, T] \to \mathbb{R}$, let $\rho(u, M) = \int_{[0,T]} M(u(t))dt$. The space $\mathcal{L}_M = \{u|u : [0, T] \to \mathbb{R},\ let\ \rho(u, M)\ is\ finite\}$ is called the Orlicz class and let L_M be the space of all equivalent classes of functions in $\mathcal{L}_M$ which are equal almost everywhere with respect to the Lebesgue measure.

Remark 2.2.

(1) *L_M is linear if and only if M satisfies the $\triangle_2$-condition.*
(2) *If u is summable then u is in L_M.*

(E) Let M and N be mutually complimentary N-functions. We let $\mathcal{V}_M = \{u \in L_M|u : [0, T] \to \mathbb{R}\ is\ measurable\ such\ that\ for\ all\ v\ in\ L_N,\ (u, v) = \int_{[0,T]} u(t)v(t)dt < +\infty\ \}$.

Let O_M be the space of all equivalent classes of functions in $\mathcal{V}_M$ which are equal almost everywhere with respect to the Lebesgue measure. From Young's inequality, we have $L_M \subset O_M$.

For u in O_M, $||u||_M = \sup_{\rho(v,N)\leq 1}(u, v)$ is called the Orlicz norm of u and $||u||_{(M)} = \inf_{k>0,\rho(u/k,M)\leq 1} k$ is called the Luxemberg norm of u.

Remark 2.3.

(1) *For u is O_M, $||u||_M \leq 1 + \rho(u, M)$.*
(2) *If M satisfies the $\triangle_2$-condition then $(O_M, ||\cdot||_M)$ is a separable Banach space and $L_M = O_M$.*
(3) *For u in O_M, $||u||_{(M)} \leq ||u||_M \leq 2||u||_{(M)}$.*
(4) *Let M and N are mutually complimentary N-functions, let E_M be the closure of L_∞ with respect to the topology generated be the norm $||\cdot||_M$ and let $V^\star$ be the dual space of the normed vector space V. Then $(E_M, ||\cdot||_{(M)})^\star = (O_N, ||\cdot||_N)$ and $(E_M, ||\cdot||_M)^\star = (O_N, ||\cdot||_{(N)})$.*
(5) *If M satisties the $\triangle_2$-condition then $E_M = L_M = O_M$. So, if M satisfies the $\triangle_2$-condition then $(O_M, ||\cdot||_M)$ is reflexive.*
(6) *Since $L_\infty \subset L_2 \subset O_M$, the closure of L_2 with respect to the topology generated by the norm $||\cdot||_M$ is O_M.*
(7) *Let M and N are mutually complimentary N-functions. For u is O_M and for v in O_N, $|(u, v)| \leq \rho(u, M) + \rho(v, N)$, $|(u, v)| \leq$*

$||u||_M||v||_{(N)}$, *and* $|(u,v)| \le ||u||_{(M)}||v||_N$. *Hence for u in* L_2, $||u||_2 \le ||u||_M \le 2||u||_{(M)}$.

(8) *M satisfies the* $\triangle_a$*-condition if and of for u in* O_M, u^2 *belongs to* O_M*[3].*

(9) *If M satisfies the* $\triangle_2$*-condition and the* $\triangle_a$*-condition, then for for u, v in* O_M*, there is a constant c such the* $||uv||_M \le c||u||_M||v||_M$.

(F) A subset I of L_2 of the form $I = \{u \in L_2 \mid P(u) \in F\}$ is called a cylinder set where P is a finite demensional orthogonal projection on L_2 and F is a Borel subset of $P(L_2)$. The Gaussian measure on L_2 is a set function of all cylinder sets defined as follows: If $I = \{u \in L_2 \mid P(u) \in F\}$ then $\mu(I) = (2\pi)^{-n/2}\int_F e^{-||t||^2/2}dt$ where n is a dimension of $P(L_2)$. Then μ is not σ-additive.

Suppose $\{e_n \mid n \in \mathbb{N}\}$ is and orthonormal basis of L_2. Let $\mu_{1,2,\cdots,n}(F) = \mu\{u \in L_2 \mid ((u,e_1),(u,e_2),\cdots,(u,e_n)) \in F\}$. Then $\{\mu_{1,2,\cdots,n}\}$ is a consistence family of probability measures. By Kolomogorov's theorem, there is a probability measure space (Ω,ω) and random variables $\xi_n : \Omega \to \mathbb{R}$ $(n \in \mathbb{N})$ such that $\omega(\{z \in \Omega|((\xi_1(z),\xi_2(z),\cdots,\xi_n(z)) \in F\})$ $= \mu_{1,2,\cdots,n}(F)$. Without loss of generality, we can put $\Omega = O_M$ because $O_M \subset L_0$, the space of all measurable functions on $[0,T]$ with the topology of convergence in measure.

Remark 2.4.

(1) O_M *is a closed subset of* L_0.

(2) *For non-zero v in* O_N *and for a real number a,* $\omega(\{u \in O_M| \ (u,v) < a\}) = \frac{1}{\sqrt{2\pi\|v\|_{(N)}}}\int_{-\infty}^{a} e^{-t^2/(2\|v\|_{(N)})}dt.$

(G) For two Borel measures m_1 and m_2, we let $m_1 * m_2(E+F) = m_1 \times m_2(E \times F)$ for E, F in $\mathcal{B}(O_M)$, the set of all Borel subsets of O_M. For $\lambda > 0$ and for B in $\mathcal{B}(O_M)$, let $\omega_\lambda(B) = \omega(\lambda^{-1/2}B)$. Then for two positive real numbers s and t, $\omega_\lambda * \omega = \omega_{\sqrt{s^2+t^2}}$ and $\omega_\lambda * \delta_0 = \omega_\lambda$, where δ_0 is the Dirac measure centered at 0.

3. The analogue of Wiener space with values in Orlicz space

Throughout this section, let M be an N-function which satisfies the $\triangle_2$-condition, let M and N be mutually complimentary N-functions, let $C(O_M)$ be the space of all continuous function defined on the interval $[0,T]$ with values in O_M in the norm $\|y\|_{C(O_M)} = \sup_{0\le t\le T}\|y(t)\|_M$ and let ϕ be a probability Borel measure on O_M.

Let $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ be given with $0 = t_0 < t_1 < t_2 < \cdots < t_n \leq T$ and let $T_{\vec{t}} : O_M^{n+1} \to O_M^{n+1}$ be a function given by

$$T_{\vec{t}}((t_0, t_1, \cdots, t_n)) = (x_0, x_0 + \sqrt{t_1}x_1, \cdots, x_0 + \sum_{j=1}^{n} \sqrt{t_j - t_{j-1}}x_j).$$

we define a set function $v_{\vec{t}}^{\phi}$ on $\mathcal{B}(O_M^{n+1})$ given by

$$v_{\vec{t}}^{\phi} = \int_{O_M} [\int_{O_M^{n+1}} (X_B \circ T_{\vec{t}})((x_0, x_1, \cdots, x_n))d(\prod_{j=1}^{n} \omega)(x_1, x_2, \cdots, x_n)]d\phi(x_0)$$

,where X_B is a characteristic function associated with B. Then $v_{\vec{t}}^{\phi}$ is a Borel measure on $(O_M^{n+1}, \mathcal{B}(O_M^{n+1}))$. Let $J_{\vec{t}} : C(O_M) \to O_M^{n+1}$ be a function with $J_{\vec{t}}(y)=((y(t_0), y(t_1), \cdots, y(t_n)))$. For Borel subset $B_0, B_1, \cdots, B_n$ in $\mathcal{B}(O_M)$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C(O_M)$ is called an interval. Let $\mathcal{J}$ be the set all such intervals. Then by (G) $\mathcal{J}$ is a semi-algebra. We define a set function M_ϕ on $\mathcal{J}$ by $M_\phi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) = v_{\vec{t}}^{\phi}(\prod_{j=0}^{n} B_j)$. Then by (G) M_ϕ is well-defined on $\mathcal{J}$, $\mathcal{B}(C(O_M))$ coincides with the smallest σ-algebra genrated by $\mathcal{J}$ and there exists a unique measure m_ϕ^M on $(C(O_M), \mathcal{B}(C(O_M)))$ such that $m_\phi^M(I) = M_\phi(I)$ for all I in $\mathcal{J}$. This measure space $(C(O_M), \mathcal{B}(C(O_M)), m_\phi^M)$ is called the analogue of Wiener measure space with values in Orlicz space.

From the change of variable theorem, we have the following two theorems.

THEOREM 3.1. *(THE WIENER INTEGRATION FORMULA 1)*
If $f : O_M^{n+1} \to \mathbb{R}$ is Borel measurable and $F : C(O_M) \to \mathbb{R}$ is a function with $F(y) = f(y(t_0), y(t_1), \cdots, y(t_n))$ then the following equality holds

$$\int_{C(O_M)} F(y)dm_\phi^M(y) = \int_{C(O_M)} f((y(t_0), y(t_1), \cdots, y(t_n))dm_\phi^M(y)$$

$$\doteq \int_{O_M} [\int_{O_M^{n+1}} (f \circ T_{\vec{t}})((x_0, x_1, \cdots, x_n))d(\prod_{j=1}^{n} \omega)(x_1, x_2, \cdots, x_n)]d\phi(x_0)$$

where $\doteq$ means that if one side exists then both sides exit and the two values are equal.

THEOREM 3.2. *(THE WIENER INTEGRATION FORMULA 2)*

If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is Borel measurable and v is a non-zero element in O_N,

$$\begin{aligned}&\int_{C(O_M)} f((y_(t_0), y_(t_1), \cdots, y(t_n))dm_\phi^M(y)\\ \doteq&\{(2\pi)^n\|v\|_{(N)}\prod_{j=1}^{n}\sqrt{t_j-t_{j-1}}\}^{-1/2}\int_{\mathbb{R}}[\int_{\mathbb{R}^{n+1}} f(s_0, s_1, \cdots, s_n)\\ &e^{-(1/2\|v\|_{(N)})\sum_{j=1}^{n}(s_j-s_{j-1})^2/(t_j-t_{j-1})}ds_n ds_{n-1}\cdots ds_1]d\phi(s_0)\end{aligned}$$

where $\doteq$ means that if one side exists, then both sides exist and the two values are equal.

Example 3.3.

(1) *Suppose $\int_{O_M}||u||_M d\phi(u)$ is finite. Then from Theorem 3.1, for $0 \le t \le T$, $F(y) = y(t)$ is m_ϕ^M-Bochner integrable on $C(O_M)$ and*

$$(B_O) - \int_{C(O)_M} y(t)dm_\phi^M(y) = (B_O) - \int_{O_M} ud\phi(u).$$

(2) *For non-zero v is O_N, for real number ξ and for $0 \le t \le T$,*

$$\int_{C(O_M)} e^{i\xi(y(t),v)}dm_\phi^M(y) = e^{i||v||_{(n)}\xi^2/2}\int_{O_M} e^{i\xi(u,v)}d\phi(u)$$

(3) *Suppose M satisfies the $\triangle_a$-codition, $0 < t_1 < t_2 \le T$ and $\int_{O_M}||u||^2{}_M d\phi(u)$ is finite. From Fernique's Theorem, we obtain $\int_{O_M}||u||_M d\omega(u)$ and $\int_{O_M}||u||_M^2 d\omega(u)$ are all finite. Hence, u, u^2 are all ω- and ϕ-Bochner integrable. Then for some positive real number c,*

$$\begin{aligned}&\int_{C(O_M)}||y(t_1)y(t_2)||_M dm_\phi^M(y) = \int_{C(O_M)} c||y(t_1)||\ ||y(t_2)||_M dm_\phi^M(y)\\ &\le c\{\int_{O_M}||u||_M d\phi(u) + (2\sqrt{t_1}+\sqrt{t_2-t_1})\int_{O_M}||u||_M d\omega(u)\\ &\quad + t_1\int_{O_M}||u||^2{}_M d\omega(u) + \sqrt{t_1}\sqrt{t_2-t_1}(\int_{O_M}||u||_M d\omega(u))^2\}\end{aligned}$$

is finite. So, the Bochner Theorem, $y(t_1)y(t_2)$ is m_ϕ^M-Bochner integrable on $C(O_M)$. Hence,

$$(B_O) - \int_{C(O_M)} y(t_1)y(t_2)dm_\phi^M(y)$$
$$= (B_O) - \int_{O_M} u^2 d\phi(u) + t_1 (B_O) - \int_{O_M} u^2 d\omega(u).$$

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